Topology and its Applications 17 (1984) 55–62
North-Holland

QUASI-MONOTONE MAPPINGS ON θ_n-CONTINUA

E.E. GRACE
Department of Mathematics, Arizona State University, Tempe, AZ 85287, USA

Eldon J. VOUGHT
Department of Mathematics, Chico State University, Chico, CA 95929, USA

Received 29 November 1982
Revised 12 April 1983

A continuous function f from a continuum X onto a continuum Y is *quasi-monotone* if, for every subcontinuum M of Y with nonvoid interior, $f^{-1}(M)$ has a finite number of components each of which is mapped onto M by f. A θ_n-continuum is one that no subcontinuum separates into more than n components. It is known that if f is quasi-monotone and X is a θ_1-continuum, then Y is a θ_1-continuum or a θ_2-continuum that is irreducible between two points. Examples are given to show that this cannot be generalized to a θ_n-continuum and $n+1$ points for any $n>1$, but it is proved that if f is quasi-monotone and X is a θ_n-continuum, then Y is a θ_n-continuum or a θ_{n+1}-continuum that is the union of $n+2$ continua $H, S_1, S_2, \ldots, S_{n+1}$, where, for each i, S_i is the closure of a component of $Y\setminus H$, S_i is irreducible from some point p_i to H, and H is irreducible about its boundary. Some theorems and examples are given concerning the preservation of decomposition elements by a quasi-monotone map defined on a θ_n-continuum that admits a monotone, upper-semicontinuous decomposition onto a finite graph.

AMS (MOS) Subj. Class. (1980): Primary 54C10, 54F20; Secondary 54B15, 54E35, 54G20

θ_n-continuum
quasi-monotone mapping
condensation decomposition
set of irreducibility
indecomposable continuum
θ'_n-continuum
ω-connected
δ-connected
T-closed set

1. Introduction

A *continuum* is a compact, connected, metric space. A *θ_n-continuum* is a continuum that is not separated into more than n components by any subcontinuum, i.e., it is a continuum that is not an $(n+1)$*-od*. A function f from a continuum X onto a continuum Y is quasi-monotone [9, p. 151] (also see [8, p. 136]), if it is continuous and, for every subcontinuum M of Y with nonvoid interior relative to Y, $f^{-1}(M)$ has a finite number of components each of which is mapped onto M by f.

In an earlier paper [3, Theorem 3, p. 114], the authors have shown that the image of a θ_1-continuum under a quasi-monotone mapping is either a θ_1-continuum or is irreducible between two points (and, hence, is a θ_2-continuum). If f is a quasi-monotone mapping on a θ_n-continuum; it is shown in this paper that $Y=f(X)$ is either a θ_n-continuum or a θ_{n+1}-continuum that is the union of $n+2$ continua H, $S_1, \ldots, S_{n+1}$, where, for each i, S_i is the closure of a component of $Y \backslash H$, S_i is irreducible from some point p_i to H, and H is irreducible about its boundary, $\beta(H)$. However, an example is given to show that only in the case $n=1$ must Y be irreducible about $n+1$ points, if it is not a θ_n-continuum. Certain restrictions on X and Y, however, do insure that Y is irreducible about $n+1$ points if it is not a θ_n-continuum.

θ_n-continua of special interest, called *θ'_n-continua*, are those that admit monotone, upper-semicontinuous decompositions onto finite graphs, where the elements of the decomposition have void interiors. In [4, Theorem 1, p. 263] the authors have characterized θ'_n-continua in terms of the aposyndetic set function T. Simple examples show that the property of being a θ'_n-continuum for some n is not preserved under quasi-monotone mappings. However, if the mapping $f: X \to Y$ is also countable to one and open or the mapping is open and Y is decomposable, then Y is a θ'_n-continuum for some n, and decomposition elements in X map onto decomposition elements in Y. Thus the map f induces a mapping on the quotient spaces. Examples are given showing that these results cannot be obtained if the conditions on the mappings are weakened in any significant way.

A continuum X is *δ-connected* [5, p. 90] if every pair of points of X is contained in a hereditarily decomposable subcontinuum of X. We say X is *ω-connected* if every pair of points of X is contained in a decomposable subcontinuum of X that is irreducible between the two points.

A subset I of a continuum X is a *set of irreducibility* of X, if there is a point x in X such that X is irreducible about $I \cup \{x\}$. See [6], where this concept was introduced, and [3] for related work.

2. The Main Theorem

Our main result about quasi-monotone mappings is the following Theorem.

Theorem 1. *If f is a quasi-monotone mapping from a θ_n-continuum X onto a continuum Y, then Y is a θ_{n+1}-continuum and is either a θ_n-continuum or, for any subcontinuum H that separates Y into $n+1$ separated sets, Y is the union of H and $n+1$ continua $S_1, \ldots, S_{n+1}$, where, for each i, S_i is the closure of a component of $Y \backslash H$, S_i is irreducible from some point p_i to H, and H is irreducible about its boundary.*

Proof. Let f be a quasi-monotone map from a θ_n-continuum X onto Y and suppose Y is not a θ_n-continuum, i.e., suppose there exists a continuum H in Y such that $Y \backslash H = D_1 \cup \cdots \cup D_{n+1}$, where $D_1, \ldots, D_{n+1}$ are mutually separated sets. For $i = 1, \ldots, n+1$, let $S_i = \bar{D}_i$. Suppose some S_i (without loss of generality, S_1) is not irreducible from any point to H. Clearly H is not a set of irreducibility of $H \cup S_1$. Then $H \cup S_1 = L \cup M$, where L and M are proper subcontinua of $H \cup S_1$ such that $H \subseteq L \cap M$ [6, Theorem 3, p. 336]. Note that each of L and M has nonvoid interior, and, hence, each of $f^{-1}(L)$ and $f^{-1}(M)$ has only finitely many components. Let $\{P_1, \ldots, P_{n(1)}\}$ be the set of components of $f^{-1}(H \cup S_1)$. For any component P of $f^{-1}(H \cup S_1)$, let $\{L_1, \ldots, L_i\}$ be the set of components of $f^{-1}(L)$ that lie in P, and let $\{M_1, \ldots, M_j\}$ be the set of components of $f^{-1}(M)$ that lie in P. Without loss of generality, assume $i \geq j$. Since L and M have nonvoid interiors and f is quasi-monotone, $f(L_l) = L$, for $l = 1, \ldots, i$, and $f(M_l) = M$, for $l = 1, \ldots, j$. No more than $j-1$ members of $\{L_1, \ldots, L_i\}$ are required to connect $M_1, \ldots,$ and M_j, hence, without loss of generality, $[\bigcup_{k=1}^{j-1} L_k] \cup [\bigcup_{k=1}^{j} M_k]$ is connected. But $L_j \backslash [\bigcup_{k=1}^{j} M_k] \neq \emptyset$, and so $U(P) = [\bigcup_{k=j}^{i} L_k] \backslash [\bigcup_{k=1}^{j} M_k] \neq 0$. But $\beta(U(P)) \subseteq [f^{-1}(\beta(H \cup S_1))] \cap P \subseteq [\bigcup_{k=1}^{j} M_k]$, so $P \backslash U(P)$ is a proper subcontinuum of P containing $\beta(U(P))$, and hence $X \backslash U(P)$ is connected. In fact, $X \backslash [\bigcup_{k=1}^{n(1)} U(P_k)]$ is connected, and $[f^{-1}(H \cup S_1)] \backslash [\bigcup_{k=1}^{n(1)} U(P_k)]$ has $n(1)$ components. For some $r < n(1)$ there are r components $R_1, \ldots, R_r$ of $X \backslash [f^{-1}(H \cup S_1)]$ such that $[\bigcup_{k=1}^{r} R_k] \cup [f^{-1}(H \cup S_1)]$ and $[\bigcup_{k=1}^{r} R_k] \cup [\bigcup_{k=1}^{n(1)} (P_k \backslash U(P_k))]$ are connected. Since X is a θ_n-continuum, $X \backslash \{[\bigcup_{k=1}^{r} R_k] \cup [\bigcup_{k=1}^{n(1)} (P_k \backslash U(P_k))]\}$ has at most n components. Since at least $n(1)$ of those components are contained in $f^{-1}(H \cup S_1)$, the set $X \backslash \{[\bigcup_{k=1}^{r} R_k] \cup [f^{-1}(H \cup S_1)]\}$ has at most $n - n(1)$ components. Hence $X \backslash [f^{-1}(H \cup S_1)]$ has at most $n - n(1) + r$ components. But $r < n(1)$ so $X \backslash [f^{-1}(H \cup S_1)]$ has at most $n-1$ components. But $f(X \backslash f^{-1}(H \cup S_1))$ has at least n components, and this contradiction establishes that S_1 is irreducible from some point to H.

Since S_1 was picked arbitrarily, it follows that, for each natural number $i \leq n+1$, there is a point p_i in $H \cup S_i$ such that S_i is irreducible from p_i to H. It follows that each D_i is connected and hence that no subcontinuum of Y separates Y into more than $n+1$ components. From this it follows that H is irreducible about its boundary, since otherwise, for a different choice of H, $Y \backslash H$ would have more than $n+1$ components.

Although it is proved in Theorem 1 that Y is a θ_{n+1}-continuum with certain irreducibility properties, if it is not a θ_n-continuum, Y need not be irreducible about $n+1$ points, or even about any finite set of points (but, in that case, neither is X [3, Theorem 1, p. 112]). This is shown in Example 1 below. However, the following are true.

Corollary 1. *If f is a quasi-monotone mapping from a θ_n-continuum X onto a ω-connected continuum Y, then Y is a θ_{n+1}-continuum and is either a θ_n-continuum or is irreducible about $n+1$ points.*

Proof. In the proof of Theorem 1, it is established that, for each natural number $i \leq n+1$, there is a point p_i such that S_i is irreducible from p_i to H. Because of this and the ω-connectedness of Y, $S_i = H_i \cup S_i'$, where H_i and S_i' are continua, $H_i \cap H \neq$

$\emptyset$, $S_i' \cap H = \emptyset$, and $p_i \in S_i' \setminus H_i$. So $H' = H \cup (\bigcup_{i=1}^{n+1} H_i)$ separates Y into separated sets $S_1' \setminus H_1, \ldots, S_{n+1}' \setminus H_{n+1}$. We can assume without loss of generality that H_i is irreducible from H to S_i', for $i = 1, \ldots, n+1$, and that $\beta(S_i') \cap \beta(S_j') = \emptyset$ for $i \neq j$.

Suppose Y is not irreducible about $\{p_1, \ldots, p_{n+1}\}$, i.e., suppose there is a proper subcontinuum Y' of Y that contains $\{p_1, \ldots, p_{n+1}\}$. Then Y' intersects H and S_i' for $i = 1, \ldots, n+1$. Hence Y' contains $\bigcup_{i=1}^{n+1} H_i$, and it follows that $Y' \cap H'$ is a continuum that contains $\beta(\bigcup_{i=1}^{n+1} H_i)$. But $Y' \cap H'$ is a proper subcontinuum of H', since Y' is a proper subcontinuum of Y and S_i' is irreducible from p_i to H', for $i = 1, \ldots, n+1$. But this contradicts the conclusion of Theorem 1, since here H' plays the role of H in that theorem and H' is not irreducible about its boundary.

Corollary 2. *If f is a quasi-monotone mapping from a δ-connected θ_n-continuum X onto a continuum Y, then Y is a θ_{n+1}-continuum and is either a θ_n-continuum or is irreducible about $n+1$ points.*

Proof. Assume Y is not a θ_n-continuum. Using the notation for Theorem 1 and the proof of Corollary 1, we need only show that each S_i is decomposable. Let i be in $\{1, \ldots, n+1\}$, and let M be a hereditarily decomposable continuum that contains a point of $f^{-1}(p_i)$ and a point of $f^{-1}(H)$. Without loss of generality assume M is irreducible between those sets. Since M is decomposable, M is the union of two proper subcontinua R and S. Since M is irreducible between $f^{-1}(p_i)$ and $f^{-1}(H)$, neither R nor S intersects both sets. Hence, neither $f(R)$ nor $f(S)$ intersects both $\{p_i\}$ and H. But $f(R \cup S)$ is a continuum that intersects both $\{p_i\}$ and H. Since S_i is irreducible from p_i to H, $f(R \cup S) \supseteq S_i$. Assume, without loss of generality, that $p_i \in f(S)$. Then $f(S)$ is a proper subcontinuum of S_i that has nonvoid interior. Hence, S_i is decomposable.

Perhaps the simplest quasi-monotone map from a θ_n-continuum onto a continuum that is not a θ_n-continuum (and hence, by Theorem 1, is a θ_{n+1}-continuum) is the mapping that folds a simple closed curve onto an arc. In that example, the hub continuum H, of Theorem 1, can be taken as an arc in the middle of the range and the spoke continua S_1 and S_2 can be taken as the arcs from H to the endpoints of the range. The points p_1 and p_2, then, are the endpoints of the range. The domain is essentially the union of two disjoint copies of the range with the points that correspond to p_1 identified and the points that correspond to p_2 identified. We will use ideas related to this example and information from Theorem 1 to construct the following example.

Example 1. For each natural number n greater than 1, a θ_n-continuum X, a continuum Y that is not a θ_n-continuum (but, by Theorem 1, is a θ_{n+1}-continuum) and is not irreducible about any finite point set, and a quasi-monotone map f from X onto Y.

Let K be the Knaster indecomposable continuum with one endpoint. Using the notation of Theorem 1, let $S_1, \ldots, S_{n+1}$ be disjoint, homeomorphic copies of K, and let $h_1, \ldots, h_{n+1}$ be homeomorphisms from K onto $S_1, \ldots, S_{n+1}$, respectively. Let $A = \{0, 1, 1/2, 1/3, \ldots\}$ and let $H = K \times A$ with the identifications described below (see Fig. 1). Let g be a homeomorphism from $\{(a_1, a_2) | (a_1, a_2) \in A \times A$ and $a_1 \leq a_2\}$ into K such that no point of the range of g is in the accessible composant of K and no two points of the range of g are in the same composant of K. One way of seeing that this is possible is given in [7, p. 305]. Extend g symmetrically to $A \times A$, i.e., let $g(a_2, a_1) = g(a_1, a_2)$ if $a_1 < a_2$. Identify the points $(g(a_1, a_2), a_1)$ and $(g(a_1, a_2), a_2)$ for all (a_1, a_2) in $A \times A$ such that $a_1 \neq a_2$.

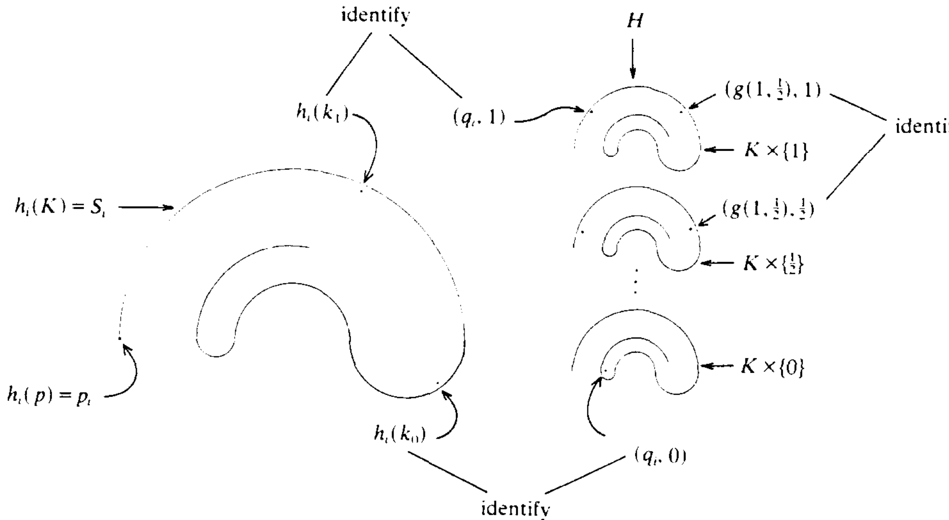

Fig. 1

Let $q_1, \ldots, q_{n+1}$ be points of K in different composants and not in any composant that intersects the range of g. Let $k_1, k_2, \ldots$ be a sequence of points in different inaccessible composants of K that converge to a point k_0 that is in yet a different inaccessible composant of K. Define Y to be the disjoint union of H and $S_1, \ldots, S_{n+1}$ with the following identifications: $(q_i, 0) = h_i(k_0)$ for $i = 1, \ldots, n+1$, and $(q_i, 1/j) = h_i(k_j)$, for $i = 1, \ldots, n+1$, and $j = 1, 2, \ldots$ (see Fig. 1). Let Y' be a homeomorphic copy of Y and let h be a homeomorphism from Y onto Y'. Let p be the endpoint of K. Let X be the disjoint union of Y and Y' with the identifications $h_i(p) = h(h_i(p))$, for $i = 1, \ldots, n+1$. Let f be the identity on Y and be h^{-1} on Y'.

2. θ_n'-Continua

Definition. A *condensation decomposition* of a continuum is a partition of the continuum into continua of condensation, i.e., into continua with void interior [2].

Definition. A θ_n-continuum X is a *θ'_n-continuum* if X admits an upper-semicontinuous, condensation decomposition $\mathscr{D}$ such that $X/\mathscr{D}$ is a finite graph.

Definition. If A is a subset of a continuum X then $T(A)=\{x\in X \mid \text{if } H \text{ is a subcontinuum of } X \text{ that contains } x \text{ in its interior then } H\cap A\neq\emptyset\}$. A set A is *T closed* if $T(A)=A$.

The next Theorem follows readily from previous work of the authors (see [4, Theorem 1, p. 263] and [2, Theorem 4(2) and (7), p. 293]).

Theorem A. *A θ_n-continuum X is a θ'_n-continuum if and only if, given a continuum of condensation H in X, it follows that $[T(H)]^0=\emptyset$, in which case, the only upper-semicontinuous, condensation decomposition of X has as its elements all of the T-closed subsets of X that contain no T-closed, proper subsets.*

The two following Theorems are easy consequences of the definitions.

Theorem 3. *If X is a continuum, f is a quasi-monotone mapping from X onto Y, and A is a T-closed subset of Y, then $f^{-1}(A)$ is T closed.*

Theorem 4. *If X is a continuum, f is an open mapping from X onto Y, and $f^{-1}(A)$ is a T-closed subset of X, then A is T closed.*

Corollary 3. *If f is an open, quasi-monotone mapping from a continuum X onto a continuum Y and $A=f^{-1}(B)$, then A is T closed iff B is T closed.*

The next Theorem follows from Theorem A and Corollary 3.

Theorem 5. *Let* (1) *X be a θ'_n-continuum and Y a θ'_m-continuum, for some integers m and n,* (2) *$\mathscr{D}$ and $\mathscr{E}$ be (the unique) upper-semicontinuous, condensation decompositions of X and Y, respectively, and* (3) *f be an open, quasi-monotone mapping from X onto Y. Then f maps elements of $\mathscr{D}$ onto elements of $\mathscr{E}$.*

Theorem 6. *If X is a θ'_n-continuum, Y is a θ'_m-continuum, for some integers m and n, and f is a quasi-monotone mapping from X onto Y, then Y is a θ'_{n+1}-continuum and either Y is a θ'_n-continuum or is irreducible about $n+1$ points.*

Proof. By Theorem 1, Y is a θ_{n+1}-continuum, and, hence, is a θ'_{n+1}-continuum. In consideration of the proof of Corollary 1, all that is needed is to show that each S_i (in the notation of Theorem 1) is decomposable. But each S_i contains an open set, and consequently, by [4, Lemma 1, p. 262] contains a set that decomposes into an arc in the canonical decomposition of Y. Hence, S_i contains a proper subcontinuum with nonvoid interior and, therefore, is decomposable.

Theorem 7. *If X is a θ'_n-continuum and f is an open, quasi-monotone mapping from X onto a decomposable continuum Y, then Y is a θ'_{n+1}-continuum, and, for upper-semicontinuous, condensation decompositions of X and Y, f maps decomposition elements onto decomposition elements.*

Proof. By Theorem 1, Y is a θ_{n+1}-continuum. If Y contains no continuum of condensation H such that $[T(H)]^0 \neq \emptyset$, then the Theorem follows by Theorem A and Theorem 5.

Assume Y contains a continuum of condensation H such that $[T(H)]^0 \neq \emptyset$. Since Y is decomposable (and is a θ_{n+1}-continuum), it is the union of two proper subcontinua R and S such that $[R \cap S]^0 = \emptyset$. Without loss of generality, $[T(H)]^0 \backslash R$ is nonvoid. Hence $[T(H \cup R)]^0 \backslash (H \cup R)$ is nonvoid, since $T(H \cup R) \supseteq T(H) \cup T(R)$ and $H^0 = \emptyset$. Since f is open, $[T(f^{-1}(H \cup R))] \backslash f^{-1}(H \cup R) \supseteq f^{-1}([T(H \cup R)] \backslash [H \cup R])$ and, hence, contains an open set. But, by [4, Lemma 1, p. 262], there is a finite collection $\{[a_1, b_1], \ldots, [a_m, b_m]\}$ of arcs in $X/\mathscr{Q}$ such that the open arc (a_i, b_i) is open in $X/\mathscr{D}$, for each i, and $\bigcup_{i=1}^{m} P^{-1}((a_i, b_i)) \subseteq f^{-1}(H \cup R) \subseteq \bigcup_{i=1}^{m} P^{-1}([a_1, b_i])$, where $\mathscr{D}$ is the upper-semicontinuous, condensation decomposition of X and P is the projection map of X onto $X/\mathscr{D}$. But $\bigcup_{i=1}^{m} P^{-1}([a_i, b_i])$ is T closed, so $[T(f^{-1}(H \cup R))] \backslash f^{-1}(H \cup R) \subseteq \bigcup_{i=1}^{m} (a_i \cup b_i)$, which is nowhere dense. This contradiction establishes that $[T(H)]^0 = \emptyset$ if H is a continuum of condensation in Y, i.e., it establishes that Y is a θ'_{n+1}-continuum, by Theorem A. By Theorem 5, f maps decomposition elements onto decomposition elements.

Theorem 8. *If X is a θ'_n-continuum and f is an open, quasi-monotone mapping onto a continuum Y such that $f^{-1}(y)$ is countable for some point y in Y, then Y is a θ'_{n+1}-continuum.*

Proof. By Theorem 7, we need only show that Y is decomposable. The set $f^{-1}(y)$ is countable and closed, so it misses $P^{-1}([a, b])$ for some arc $[a, b]$ in $X/\mathscr{Q}$, where $\mathscr{D}$ is the upper-semicontinuous, condensation decomposition of X and P is the projection map of X onto $X/\mathscr{D}$. The interior of $P^{-1}([a, b])$ is nonvoid, so $H = f(P^{-1}([a, b]))$ is a continuum with nonvoid interior, since f is open. But H is a proper subcontinuum of Y, since $y \notin H$. Hence Y is decomposable.

The next example shows that the condition that f be open cannot be dropped from Theorem 5 or Theorem 7 even if the condition that f be finite to one is added.

Example 2. An at most two-to-one, quasi-monotone mapping from a θ'_2-continuum X onto a θ'_2-continuum Y where the images of some (singleton) T-closed sets that are elements of the upper-semicontinuous, condensation decomposition of X are not T closed and hence are not decomposition elements of the upper-semicontinuous, condensation decomposition of Y.

Let K be the Knaster indecomposable continuum with one endpoint, embedded in the plane. Let Y be the union of K and a ray R, directly beneath the composant with one end-point, spiraling up onto K in such a way that no vertical line intersects R in more than one point. Let X be the union of Y and the mirror image of R in the x, y-plane. Let f be the identity function on Y (as a subset of X) and carry each point of the upper ray onto the point of K directly below it. Then K is the only nondegenerate element in the upper-semicontinuous, condensation decomposition of X or Y, and, of course, the points of the upper ray map into it but not onto it.

The condition that f be open also can not be dropped from Theorem 8, as can be seen by considering the mapping $f|[X \backslash R]$ in Example 2.

The final example shows that the countable-to-one condition can not be dropped from the hypothesis of Theorem 8, even though it can be replaced by the hypothesis that Y is decomposable, as Theorem 7 shows.

Example 3. An open, quasi-monotone map f from a θ'_2-continuum X onto an indecomposable continuum Y.

Let p and q be points of different composants of an indecomposable continuum Y. Let C be the standard, 'middle-third' Cantor set on the unit interval. Let X be the quotient space of $Y \times C$ gotten by identifying, for each gap (c, c') in the Cantor set, (p, c) with (p, c'), if $c' - c = 1/(3^n)$, for some odd integer n, and (q, c) with (q, c') otherwise. For each x in X, let $f(x) = P(Q^{-1}(x))$, where Q is the quotient map from $Y \times C$ onto X, and P is the projection map from $Y \times C$ onto Y.

References

[1] R.W. FitzGerald, Connected sets with a finite disconnection property, in: N.M. Stavrakas and K.R. Allen, eds., Studies in Topology (Academic Press, New York, 1975) pp. 139–173.

[2] E.E. Grace, Monotone decompositions of θ-continua, Trans. Am. Math. Soc. 275 (1983) 287–295.

[3] E.E. Grace and E.J. Vought, Quasi-monotone images of certain classes of continua, Gen. Topology Appl. 9 (1978) 111–116.

[4] E.E. Grace and E.J. Vought, Monotone decompositions of θ_n-continua, Trans. Am. Math. Soc. 263 (1981) 261–270.

[5] B. Knaster and S. Masurkiewicz, Sur un problème concernant les transformations continues, Fund. Math. 21 (1933) 85–90.

[6] T. Maćkowiak, Sets of irreducibility and mappings, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976) 335–339.

[7] S. Mazurkiewicz, Sur les continus indecomposables, Fund. Math. 10 (1927) 305–310.

[8] A.D. Wallace, Quasi-monotone transformations, Duke Math. J. 7 (1940) 136–145.

[9] G.T. Whyburn, Analytic Topology (Amer. Math. Soc. Colloquium Publications, 28, 1942).